

INTERSECTIONS OF SETS OF DISTANCES

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ABSTRACT. We isolate conditions on the relative asymptotic size of sets of natural numbers A, B that guarantee a nonempty intersection of the corresponding sets of distances. Such conditions apply to a large class of zero density sets. We also show that a variant of *Khintchine's Recurrence Theorem* holds for all infinite sets $A = \{a_1 < a_2 < \dots\}$ where $a_n \ll n^{3/2}$.

1. INTRODUCTION

It is a well-known phenomenon that if a set of natural numbers A has positive upper asymptotic density $\bar{d}(A) > 0$, then its *set of distances* (or *Delta-set*)

$$\Delta(A) = \{a - a' \mid a, a' \in A, a > a'\}$$

has a really rich combinatorial structure. An old problem attributed to Paul Erdős was whether the distance sets of two sets of positive upper density must necessarily meet.

- Does $\Delta(A) \cap \Delta(B) \neq \emptyset$ whenever $\bar{d}(A), \bar{d}(B) > 0$?

The answer was shortly shown to be positive, and in fact the following much stronger intersection property holds:

- If the upper density $\bar{d}(A) = \alpha > 0$ is positive, then $\Delta(A) \cap \Delta(B) \neq \emptyset$ whenever the set B contains more than $1/\alpha$ -many elements.

The proof consists of a straightforward application of the *pigeonhole principle*. The key observation is that if one takes distinct elements b_1, \dots, b_N with $N > 1/\alpha$, then the shifted sets $A + b_i$ cannot be pairwise disjoint, as otherwise $\bar{d}(\bigcup_{i=1}^N A + b_i) = \sum_{i=1}^N \bar{d}(A + b_i) = N \cdot \bar{d}(A) > 1$. The argument is then completed by noticing that $(A + b) \cap (A + b') \neq \emptyset$ for some $b \neq b'$ in B if and only if $\Delta(A) \cap \Delta(B) \neq \emptyset$.

In the last forty years, the research on the combinatorial properties of distance sets and difference sets¹ has produced many interesting results (see, e.g., [10, 5, 11, 13, 14, 15, 16, 1, 9, 3, 7, 12, 8, 4]) which are almost always grounded on the hypothesis of positive density. In this paper, we look for general properties that include the zero density case, and investigate the

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¹ By a *difference set* is meant a set of the form $A - B = \{a - b \mid a \in A, b \in B\}$. So, the set of distances $\Delta(A)$ is the positive part of $A - A$.

size of intersections $\Delta(A) \cap \Delta(B)$ depending on the relative density of A with respect to B . More generally, for $k \in \mathbb{N}$, we will consider intersections $R_k(A) \cap \Delta(B)$ where

$$R_k(A) = \{x \in \mathbb{N} : |A \cap (A + x)| \geq k\}$$

is the k -recursion set of A . Elements of $R_k(A)$ are those natural numbers that are the distance of at least k -many different pairs of elements in A ; in particular, $R_1(A) = \Delta(A)$.

The main results presented here (see Corollaries 3.5 and 4.3) can be summarized as follows.

Main Theorem. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and let $\vartheta : \mathbb{N} \rightarrow \mathbb{R}^+$ be such that $\limsup_{n \rightarrow \infty} \frac{a_n}{n \cdot \vartheta(n)} < \infty$.*

- (1) *If $\lim_{n \rightarrow \infty} \frac{\vartheta(b_n)}{n} = 0$, then $R_k(A) \cap \Delta(B)$ is infinite for all k .*
- (2) *If $\lim_{n \rightarrow \infty} \frac{\vartheta(n \cdot b_n)}{n} = 0$, then there exists a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ of elements of $\Delta(B)$ such that*

$$\limsup_{n \rightarrow \infty} \left(\frac{\frac{|A \cap (A + x_n) \cap [1, n]|}{n}}{\left(\frac{|A \cap [1, n]|}{n} \right)^2} \right) \geq 1.$$

We remark that the above results apply to a large class of zero density sets; e.g., when $B = \mathbb{N}$, (1) applies whenever $a_n \lll n^2$, and (2) applies whenever $a_n \lll n^{3/2}$. By way of examples, we list below three consequences (see Examples 3.7, 3.9, and 4.5).

Example 1. If $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$ and $B = \{b_n\}$ is such that $\log b_n \lll n^{1-\varepsilon}$ for some $\varepsilon > 0$, then the intersections $R_k(A) \cap \Delta(B)$ are infinite for all k .

Example 2. Let $A = \{\lfloor K \cdot n \sqrt{n} \rfloor\}$ and $B = \{M \cdot n^3\}$. If $K^2 \cdot M < \frac{4}{27}$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k .

Example 3. Let $A = \{a_n\}$ have the same asymptotic size as the set of prime numbers, i.e. $\lim_{n \rightarrow \infty} \frac{a_n}{n \cdot \log n} = 1$, and assume that $B = \{b_n\}$ is sub-exponential, i.e. $\log b_n \lll n$. Then for every $\varepsilon > 0$ there exist infinitely many n and elements $x_n \in \Delta(B)$ such that

$$|A \cap (A + x_n) \cap [1, a_n]| \geq \frac{n}{\log n} \cdot (1 - \varepsilon).$$

Of course, the asymptotic conditions considered in our theorems about sequences $A = \{a_n\}$ may be reformulated by using the corresponding counting functions $A(u) = |\{a \in A \mid a \leq u\}|$. For instance, the role of a_n/n is played by $u/A(u)$, and so forth.

Notation. The natural numbers \mathbb{N} are the set of *positive* integers. Letters $n, m, h, k, s, t, \nu, \mu, N$ will be used for natural numbers, and upper-case letters A, B, C , will be used for sets of natural numbers. For infinite sets

$A \subseteq \mathbb{N}$ we use the brace notation $A = \{a_n\}$ to mean that elements a_n are arranged in increasing order:

$$A = \{a_n\} = \{a_1 < a_2 < \dots < a_n < \dots\}.$$

We write $A + x = \{a + x \mid a \in A\}$ to denote the *shift* of A by x . For functions $f : \mathbb{N} \rightarrow \mathbb{R}^+$ taking positive real values, we write $a_n \lll f(n)$ to mean that $\lim_{n \rightarrow \infty} a_n/f(n) = 0$.² By $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$ is denoted the *integer part* of a real number x . Finally, recall the notion of *upper asymptotic density* $\overline{d}(A)$ for sets $A \subseteq \mathbb{N}$:

$$\overline{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap [1, n]|}{n}.$$

2. PRELIMINARY RESULTS

Let us start with a straightforward consequence of the *pigeonhole principle*.

Proposition 2.1. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers. If there exist N, ν such that $a_N + b_\nu \leq N \cdot \nu$ then $\Delta(A) \cap \Delta(B) \neq \emptyset$. In particular, if $\liminf_{n \rightarrow \infty} \frac{a_n + b_n}{n^2} < 1$ then $\Delta(A) \cap \Delta(B) \neq \emptyset$.*

Proof. Fix N, ν as in the hypothesis. The sumset

$$\{a_i + b_j \mid 1 \leq i \leq N; 1 \leq j \leq \nu\} \subseteq [2, a_N + b_\nu] \subseteq [2, N \cdot \nu]$$

contains at most $N \cdot \nu - 1$ elements. So, by the *pigeonhole principle*, there exist $(i, j) \neq (i', j')$ such that $a_i + b_j = a_{i'} + b_{j'}$. Clearly $i \neq i'$, say $i > i'$. Then $a_i - a_{i'} = b_{j'} - b_j \in \Delta(A) \cap \Delta(B) \neq \emptyset$. If $\liminf_n \frac{a_n + b_n}{n^2} < 1$, pick N such that $\frac{a_N + b_N}{N^2} < 1$, and apply the above argument with $N = \nu$. \square

Remark 2.2. The above result is best possible because there exist infinite sets $A = \{a_n\}$ and $B = \{b_n\}$ such that $\liminf_n \frac{a_n + b_n}{n^2} = 1$ but $\Delta(A) \cap \Delta(B) = \emptyset$. The following example is due to P. Erdős and R. Freud [6].

- Let A be the set of all natural numbers that are sums of even powers of 2, including $1 = 2^0$.
- Let B be the set of all natural numbers that are sums of odd powers of 2.

It only takes a little computation to verify that:

- $b_n = 2 \cdot a_n$ for all n ;
- $\liminf_{n \rightarrow \infty} a_n/n^2 = 1/3$ is attained on the subsequence $n_k = 2^k - 1$;
- $\liminf_{n \rightarrow \infty} \frac{a_n + b_n}{n^2} = 1$.

Besides, an equality $a_i - a_j = b_s - b_t \Leftrightarrow a_i + b_t = a_j + b_s$ holds if and only if $i = j$ and $s = t$, since every natural number is uniquely written as a sum of powers of 2. It follows that $\Delta(A) \cap \Delta(B) = \emptyset$.

² This is equivalent to Landau notation $a_n = o(f(n))$.

In order to improve on the previous result, we will use the following elementary inequality.

Lemma 2.3. *Let $A = \{a_1 < \dots < a_N\}$ and $B = \{b_1 < \dots < b_\nu\}$ be finite sets of natural numbers. For every $h \leq \nu/2$ there exists $x \in \Delta(B)$ such that $x \geq h$ and*

$$|A \cap (A + x)| \geq \frac{N^2}{a_N + b_\nu} - \frac{N \cdot (2h - 1)}{\nu} = \frac{N^2}{a_N} \cdot \frac{1 - \frac{(a_N + b_\nu)(2h - 1)}{N \cdot \nu}}{1 + \frac{b_\nu}{a_N}}.$$

The above inequality is strict except when $h = 1$ and $N \cdot \nu = a_N + b_\nu$.

Proof. Let us first consider the case $h = 1$. Let I be the interval $[1, a_N + b_\nu]$, and for every $i = 1, \dots, \nu$, let $\chi_i : I \rightarrow \{0, 1\}$ be the characteristic function of the shifted sets $A + b_i \subseteq I$. Notice that

$$\sum_{x \in I} \left(\sum_{i=1}^{\nu} \chi_i(x) \right) = \sum_{i=1}^{\nu} \left(\sum_{x \in I} \chi_i(x) \right) = \sum_{i=1}^{\nu} |A + b_i| = N \cdot \nu.$$

By *Cauchy-Schwartz inequality*, we obtain:

$$\begin{aligned} N^2 \cdot \nu^2 &= \left(\sum_{x \in I} 1 \cdot \left(\sum_{i=1}^{\nu} \chi_i(x) \right) \right)^2 \leq \left(\sum_{x \in I} 1^2 \right) \cdot \sum_{x \in I} \left(\sum_{i=1}^{\nu} \chi_i(x) \right)^2 \\ &= |I| \cdot \sum_{x \in I} \left(\sum_{i,j=1}^{\nu} \chi_i(x) \cdot \chi_j(x) \right) = |I| \cdot \sum_{i,j=1}^{\nu} \left(\sum_{x \in I} \chi_i(x) \cdot \chi_j(x) \right) \\ &= (a_N + b_\nu) \cdot \sum_{i,j=1}^{\nu} |(A + b_i) \cap (A + b_j)|. \end{aligned}$$

If $M = \max\{|(A + b_i) \cap (A + b_j)| : 1 \leq i < j \leq \nu\}$ then

$$\begin{aligned} \sum_{i,j=1}^{\nu} |(A + b_i) \cap (A + b_j)| &= \sum_{i=1}^{\nu} |A + b_i| + 2 \cdot \sum_{1 \leq i < j \leq \nu} |(A + b_i) \cap (A + b_j)| \\ &\leq N \cdot \nu + 2 \cdot \binom{\nu}{2} \cdot M = \nu \cdot (N + (\nu - 1) \cdot M) \end{aligned}$$

(The above expressions are well-defined because we are assuming $h \leq \nu/2$, and hence $\nu \geq 2$.) By combining with the previous inequalities, we get that

$$N^2 \cdot \nu \leq (a_N + b_\nu) \cdot (N + (\nu - 1) \cdot M),$$

and hence

$$M \geq \frac{\nu}{\nu - 1} \cdot \frac{N^2}{a_N + b_\nu} - \frac{N}{\nu - 1} = \frac{\nu}{\nu - 1} \cdot \left(\frac{N^2}{a_N + b_\nu} - \frac{N}{\nu} \right) \geq \frac{N^2}{a_N + b_\nu} - \frac{N}{\nu}.$$

Notice that the last inequality is strict provided that $\frac{N^2}{a_N + b_\nu} - \frac{N}{\nu} > 0$ or, equivalently, when $N \cdot \nu > a_N + b_\nu$. Notice also that, since $M \geq 0$, the strict inequality trivially holds also when $N \cdot \nu < a_N + b_\nu$. Observe that if

$M = |(A + b_s) \cap (A + b_t)|$, then $M = |A \cap (A + x)|$ where $x = b_s - b_t \in \Delta(B)$, and this completes the proof of the case $h = 1$.

Now let $h \geq 2$. Let μ be such that $\mu h \leq \nu < (\mu + 1)h$, and consider the set $B' = \{b'_1 < \dots < b'_\mu\} \subset B$ where $b'_i = b_{ih}$. Notice that $\mu \geq 2$, because we are assuming $h \leq \nu/2$, and so we can apply the property proved above to prove the existence of an element $x \in \Delta(B')$ such that

$$|A \cap (A + x)| \geq \frac{N^2}{a_N + b'_\mu} - \frac{N}{\mu} \geq \frac{N^2}{a_N + b_\nu} - \frac{N}{\mu}.$$

For suitable indexes $1 \leq s < t \leq \mu$, one has that $x = b'_t - b'_s = b_{th} - b_{sh} \geq th - ts \geq h$. Finally, notice that $\frac{N}{\mu} = \frac{N}{\nu} \cdot \frac{\nu}{\mu} < \frac{N}{\nu} \cdot \frac{(\mu+1)h}{\mu}$, and since $\frac{(\mu+1)h}{\mu} \leq 2h - 1$, the thesis follows. Indeed, $\mu h + h \leq 2\mu h - \mu \Leftrightarrow \mu h \geq \mu + h$, and the last inequality holds because $\mu, h \geq 2$. \square

Theorem 2.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that*

$$\liminf_{n \rightarrow \infty} \frac{a_n + b_n}{n^2} = 0.$$

Then $R_k(A) \cap \Delta(B)$ is infinite for all k .³

Proof. Fix an arbitrary $h \in \mathbb{N}$. For every $n \geq 2h$, apply Lemma 2.3 to the finite sets $A_n = \{a_1 < \dots < a_n\}$ and $B_n = \{b_1 < \dots < b_n\}$, and get the existence of an element $x_n \in \Delta(B_n)$ such that $x_n \geq h$ and

$$|A \cap (A + x_n) \cap [1, a_n + b_n]| \geq |A_n \cap (A_n + x_n)| > \frac{n^2}{a_n + b_n} - (2h - 1).$$

By the hypothesis, the sequence on the right side is unbounded as n goes to infinity and so, for every k , there exists $x_n \in \Delta(B_n) \subseteq \Delta(B)$ with $x_n \geq h$ and $|A \cap (A + x_n)| \geq k$. As h was arbitrary, this proves that intersections $R_k(A) \cap \Delta(B)$ are infinite. \square

Next, we prove that when A has positive asymptotic density, the set of all possible shifts x that yield “large” intersections $A \cap (A + x)$ is “combinatorially large”, in the sense that it meets all sufficiently large Delta-sets.

Theorem 2.5. *Let A be a set of natural numbers with $\bar{d}(A) = \alpha > 0$. Then for every $\varepsilon > 0$ and for every set B with $|B| \geq \alpha/\varepsilon$, one has*

$$\{x \mid \bar{d}(A \cap (A + x)) \geq \alpha^2 - \varepsilon\} \cap \Delta(B) \neq \emptyset.$$

Proof. Notice first that the limit superior for the upper asymptotic density is attained along intervals of the form $[1, a_n]$; so, by passing to a subsequence if necessary, we can directly assume that $\lim_{n \rightarrow \infty} n/a_n = \alpha$. Without loss of generality, let us assume that $B = \{b_1 < \dots < b_\nu\}$ is finite with $\nu \geq \alpha/\varepsilon$.

³ Recall that $R_k(A) = \{x \in \mathbb{N} : |A \cap (A + x)| \geq k\}$.

For every n , apply Lemma 2.3 to the finite sets $A_n = \{a_1 < \dots < a_n\}$ and B (with $h = 1$) and obtain the existence of an element $x_n \in \Delta(B)$ such that

$$\begin{aligned} |A \cap (A + x_n) \cap [1, a_n]| &\geq |A \cap (A + x_n) \cap [1, a_n + b_\nu]| - b_\nu \geq \\ &\geq |A_n \cap (A_n + x_n)| - b_\nu \geq \frac{n^2}{a_n} \cdot \frac{1 - \frac{a_n + b_\nu}{n \cdot \nu}}{1 + \frac{b_\nu}{a_n}} - b_\nu. \end{aligned}$$

Since ν is fixed, by passing to the limit as n goes to infinity, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|A \cap (A + x_n) \cap [1, a_n]|}{a_n} &\geq \\ &\geq \lim_{n \rightarrow \infty} \frac{n^2}{a_n^2} \cdot \frac{1 - \frac{a_n}{n\nu} - \frac{b_\nu}{n\nu}}{1 + \frac{b_\nu}{a_n}} - \frac{b_\nu}{a_n} = \alpha^2 \cdot \left(1 - \frac{1}{\alpha \cdot \nu}\right) \geq \alpha^2 - \varepsilon. \end{aligned}$$

Now notice that the sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ takes values in the finite set $\Delta(B)$, and so there exists an element $x \in \Delta(B)$ such that the limit superior is attained along a subsequence $\{n_k\}$ where $x_{n_k} = x$ for all k . Such an element x yields the thesis because

$$\begin{aligned} \overline{d}(A \cap (A + x)) &\geq \limsup_{k \rightarrow \infty} \frac{|A \cap (A + x) \cap [1, a_{n_k}]|}{a_{n_k}} = \\ &= \limsup_{k \rightarrow \infty} \frac{|A \cap (A + x_{n_k}) \cap [1, a_{n_k}]|}{a_{n_k}} \geq \alpha^2 - \varepsilon. \end{aligned}$$

□

As a straight corollary, we obtain the well-known density version of *Khintchine's Recurrence Theorem* for sets of integers (see, e.g., §5 of [2]).

Corollary 2.6. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers. If $\overline{d}(A) > 0$ then for every $\varepsilon > 0$ the following intersection is infinite:*

$$\{x \mid \overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon\} \cap \Delta(B).$$

In consequence:

- (1) *All intersections $R_k(A) \cap \Delta(B)$ are infinite;*
- (2) $\limsup_{x \in \Delta(B)} \frac{\overline{d}(A \cap (A + x))}{\overline{d}(A)^2} \geq 1.$

Proof. For every h , by applying the previous theorem to A and $B^h = \{b_{hn}\}$, one gets the existence of an element $x_h = b_{hs} - b_{ht} \in \Delta(B^h) \subseteq \Delta(B)$ with $\overline{d}(A \cap (A + x_h)) \geq \overline{d}(A)^2 - \varepsilon$. Notice that $x_h \geq hs - ht \geq h$. This proves that there are arbitrarily large elements in the intersection $\{x \mid \overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon\} \cap \Delta(B)$, as desired.

(1). Every set of positive upper density is infinite, and so, for every k , the set $\{x \mid \overline{d}(A \cap (A + x)) > \overline{d}(A)^2 - \varepsilon\} \subseteq R_k(A)$ whenever $0 < \varepsilon < \overline{d}(A)^2$.

(2). By what proved above, for every $\varepsilon > 0$ there are infinitely many elements $x \in \Delta(B)$ such that $\overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon$; but then

$\limsup_{x \in \Delta(B)} \overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon$. Since $\varepsilon > 0$ can be taken arbitrarily small, the thesis follows. \square

Further on in this paper, we will show that a similar result as (2) can be proved for a large class of zero density sets (see Corollary 4.3).

3. INTERSECTION PROPERTIES

We saw in Theorem 2.4 that $\Delta(A) \cap \Delta(B) \neq \emptyset$ whenever both A and B are asymptotically larger than the set of squares. We now sharpen that result, and prove a general intersection property that also applies when b_n/n^2 goes to infinity.

Theorem 3.1. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers where $a_n \lll n^2$. Denote by $f(n) = a_n/n$ and by $g(n) = b_n/n$.*

(1) *If there exists a constant $c > 1$ such that*

$$\liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} < 1 - \frac{1}{c}$$

then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k .

(2) *If for arbitrarily large constants c one has*

$$\liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} = 0$$

then $R_k(A) \cap \Delta(B)$ is infinite for all k .

(3) *If there exists a constant $\varepsilon > 0$ such that*

$$\liminf_{n \rightarrow \infty} \frac{f(\lfloor \varepsilon \cdot b_n \rfloor)}{n} < 1$$

then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k .

(4) *If there exists a constant $\varepsilon > 0$ such that*

$$\liminf_{n \rightarrow \infty} \frac{f(\lfloor \varepsilon \cdot b_n \rfloor)}{n} = 0$$

then $R_k(A) \cap \Delta(B)$ is infinite for all k .

Proof. In the following, without loss of generality, we will always assume that $n \lll a_n$. Indeed, $n \lll a_n$ fails if and only if the upper asymptotic density $\overline{d}(A)$ is positive, and in this case the four properties above are all proved by Corollary 2.6.

(1). Let

$$\liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} = l < 1 - \frac{1}{c}.$$

For every n , let $\tau(n) = \lfloor c \cdot f(n) \rfloor$, and apply Lemma 2.3 with $h = 1$ to the sets $A_n = \{a_1 < \dots < a_n\}$ and $B_{\tau(n)} = \{b_1 < \dots < b_{\tau(n)}\}$. We obtain the existence of an element $x_n \in \Delta(B_{\tau(n)}) \subseteq \Delta(B)$ such that:

$$|A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| \geq |A_n \cap (A_n + x_n)| \geq \frac{n^2}{a_n} \cdot \frac{1 - \frac{a_n + b_{\tau(n)}}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}}.$$

Since we are assuming $n \lll a_n$, we have that $\lim_{n \rightarrow \infty} f(n) = \infty$, and so

$$\lim_{n \rightarrow \infty} \frac{a_n}{n \cdot \tau(n)} = \lim_{n \rightarrow \infty} \frac{f(n)}{\lfloor c \cdot f(n) \rfloor} = \frac{1}{c}.$$

Besides,

$$\liminf_{n \rightarrow \infty} \frac{b_{\tau(n)}}{n \cdot \tau(n)} = \liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} = l,$$

and

$$\liminf_{n \rightarrow \infty} \frac{b_{\tau(n)}}{a_n} = \liminf_{n \rightarrow \infty} \frac{\lfloor c \cdot f(n) \rfloor}{f(n)} \cdot \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} = c \cdot l.$$

Notice that the two limit inferiors above are attained along the same subsequence, and so

$$\limsup_{n \rightarrow \infty} \frac{1 - \frac{a_n + b_{\tau(n)}}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}} = \frac{1 - (\frac{1}{c} + l)}{1 + c \cdot l} > 0.$$

By using the hypothesis $a_n \lll n^2$, i.e. $\lim_{n \rightarrow \infty} n^2/a_n = \infty$, we can then conclude that

$$\limsup_{n \rightarrow \infty} |A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| = \infty.$$

This shows that for every k one finds elements $x_n \in \Delta(B)$ such that $|A \cap (A + x_n)| \geq k$, and hence $R_k(A) \cap \Delta(B) \neq \emptyset$.⁴

(2). Fix $h > 1$. For every n , let $\tau(n) = \lfloor 2h \cdot f(n) \rfloor$, and apply Lemma 2.3 to the sets A_n and $B_{\tau(n)}$ so as to get the existence of an element $x_n \in \Delta(B_{\tau(n)}) \subseteq \Delta(B)$ such that $x_n \geq h$ and

$$|A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| \geq \frac{n^2}{a_n} \cdot \frac{1 - \frac{(a_n + b_{\tau(n)})(2h-1)}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}}.$$

Now use the same arguments as in the proof of the previous property (1). Since in our case $c = 2h$ and $l = 0$, we obtain that

$$\limsup_{n \rightarrow \infty} \frac{1 - \frac{(a_n + b_{\tau(n)})(2h-1)}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}} = \frac{1 - (\frac{1}{c} + l)(2h-1)}{1 + c \cdot l} = 1 - \frac{2h-1}{2h} > 0.$$

⁴ We remark that the map $n \mapsto x_n$ may not be 1-1, and so the above argument *does not* prove that $R_k(A) \cap \Delta(B)$ contains infinitely many elements.

By the hypothesis $a_n \ll n^2$, we conclude that

$$\limsup_{n \rightarrow \infty} |A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| = \infty.$$

So, for every k , there exist elements $x_n \in \Delta(B_n) \subseteq \Delta(B)$ such that $x_n \geq h$ and $|A \cap (A + x_n)| \geq k$. Since h is arbitrary, this shows that the intersection $R_k(A) \cap \Delta(B)$ is infinite, as desired.

(3). The proof is entirely similar to the proof of (1), by applying Lemma 2.3 to the sets $A_{\sigma(n)}$ and B_n where $\sigma(n) = \lfloor \varepsilon \cdot b_n \rfloor$. Indeed, notice that

$$\liminf_{n \rightarrow \infty} \frac{a_{\sigma(n)}}{\sigma(n) \cdot n} = \liminf_{n \rightarrow \infty} \frac{f(\lfloor \varepsilon \cdot b_n \rfloor)}{n} = l < 1.$$

Besides,

$$\lim_{n \rightarrow \infty} \frac{b_n}{\sigma(n)} = \lim_{n \rightarrow \infty} \frac{b_n}{\lfloor \varepsilon \cdot b_n \rfloor} = \frac{1}{\varepsilon} < \infty,$$

and so

$$\lim_{n \rightarrow \infty} \frac{b_n}{\sigma(n) \cdot n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n}{a_{\sigma(n)}} = \lim_{n \rightarrow \infty} \frac{b_n}{\sigma(n)} \cdot \frac{\sigma(n)}{a_{\sigma(n)}} = 0.$$

Thus we have the existence of elements $x_n \in \Delta(B)$ such that

$$\limsup_{n \rightarrow \infty} |A \cap (A + x_n) \cap [1, a_{\sigma(n)} + b_n]| \geq \limsup_{n \rightarrow \infty} \frac{\sigma(n)^2}{a_{\sigma(n)}} \cdot \frac{1 - \frac{a_{\sigma(n)} + b_n}{\sigma(n) \cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} = \infty,$$

and the thesis follows.

(4). For fixed $h > 1$, we proceed as in (3) and obtain the existence of elements $x_n \in \Delta(B_n) \subseteq \Delta(B)$ with $x_n \geq h$ and such that

$$\limsup_{n \rightarrow \infty} |A \cap (A + x_n) \cap [1, a_{\sigma(n)} + b_n]| \geq \limsup_{n \rightarrow \infty} \frac{\sigma(n)^2}{a_{\sigma(n)}} \cdot \frac{1 - \frac{(a_{\sigma(n)} + b_n)(2h-1)}{\sigma(n) \cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}}.$$

As we are assuming $l = 0$, the above limit superior is infinite. Finally, since h can be taken arbitrarily large, the thesis follows. \square

Remark 3.2. Under the (mild) hypothesis that $g(n)$ be non-decreasing, one can prove (3) and (4) as consequences of (1) and (2), which are therefore basically stronger properties. Indeed, given $\varepsilon > 0$, let us assume that $\tau(n) = f(\lfloor \varepsilon \cdot b_n \rfloor)/n$ satisfies the condition $\liminf_{n \rightarrow \infty} \tau(n) = l < 1$. Then for every constant c such that $c \cdot l < 1$, we have that $c \cdot \tau(n) \cdot n < n$ for infinitely many n , and so

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} &\leq \liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(\lfloor \varepsilon \cdot b_n \rfloor) \rfloor)}{\lfloor \varepsilon \cdot b_n \rfloor} = \\ &= \liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot \tau(n) \cdot n \rfloor)}{\lfloor \varepsilon \cdot n \cdot g(n) \rfloor} = \frac{1}{\varepsilon} \cdot \liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot \tau(n) \cdot n \rfloor)}{n \cdot g(n)} \leq \\ &\leq \frac{1}{\varepsilon} \cdot \liminf_{n \rightarrow \infty} \frac{g(n)}{n \cdot g(n)} = 0. \end{aligned}$$

Notice that, since $l < 1$, we can pick constants $c > 1$ such that $c \cdot l < 1$, and this completes the proof of (1) \Rightarrow (3). Besides, if $l = 0$, every constant $c > 1$ trivially satisfies $c \cdot l < 1$, and also (2) \Rightarrow (4) follows.

As a consequence of the previous theorem, one can isolate a large class of sets B such that $R_k(A) \cap \Delta(B) \neq \emptyset$, in terms of their density relative to A .

Corollary 3.3. *Let $A = \{a_n = n \cdot f(n)\}$ be an infinite set of natural numbers where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing unbounded function, and assume that the infinite set of natural numbers $B = \{b_n\}$ is such that*

$$\lim_{n \rightarrow \infty} \frac{b_n/n}{f^{-1}(\varepsilon \cdot n)} = 0 \quad \text{for all } \varepsilon > 0.$$

Then intersections $R_k(A) \cap \Delta(B)$ are infinite for all k .

Proof. Fix $c > 1$, and let $\tau(n) = \lfloor c \cdot f(n) \rfloor$ and $\varepsilon = 1/c$. Then $f^{-1}(\varepsilon \cdot \tau(n)) \leq n$ and

$$0 \leq \lim_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \leq \lim_{n \rightarrow \infty} \frac{g(\tau(n))}{f^{-1}(\varepsilon \cdot \tau(n))} = \lim_{n \rightarrow \infty} \frac{b_{\tau(n)}/\tau(n)}{f^{-1}(\varepsilon \cdot \tau(n))} = 0.$$

Thus (2) of the previous Theorem applies, and we get the thesis. \square

When $\varepsilon = 1$, items (3) and (4) in Theorem 3.1 have the advantage that can be reformulated in the following simpler form:

Corollary 3.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers where $a_n \ll n^2$, and let*

$$\liminf_{n \rightarrow \infty} \frac{a_{b_n}}{n \cdot b_n} = l.$$

If $l < 1$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k ; and if $l = 0$ then $R_k(A) \cap \Delta(B)$ is infinite for all k .

A consequence that is easily applied in several examples is the following:

Corollary 3.5. *Given a function $\vartheta : \mathbb{N} \rightarrow \mathbb{R}^+$ and infinite sets of natural numbers $A = \{a_n\}$ and $B = \{b_n\}$, denote by:*

$$\liminf_{n \rightarrow \infty} \frac{a_n}{n \cdot \vartheta(n)} = \underline{\ell}; \quad \limsup_{n \rightarrow \infty} \frac{a_n}{n \cdot \vartheta(n)} = \bar{\ell};$$

$$\liminf_{n \rightarrow \infty} \frac{\vartheta(b_n)}{n} = \underline{\ell}'; \quad \limsup_{n \rightarrow \infty} \frac{\vartheta(b_n)}{n} = \bar{\ell}'.$$

If $\underline{\ell} \cdot \bar{\ell}' < 1$ or $\bar{\ell} \cdot \underline{\ell}' < 1$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k ; and if $\underline{\ell} \cdot \bar{\ell}' = 0$ or $\bar{\ell} \cdot \underline{\ell}' = 0$ then $R_k(A) \cap \Delta(B)$ is infinite for all k .⁵

⁵ By writing $\underline{\ell} \cdot \bar{\ell}' < 1$ or $\bar{\ell} \cdot \underline{\ell}' = 0$, it is implicitly assumed that both $\underline{\ell}$ and $\bar{\ell}'$ are finite; and similarly in the other cases.

Proof. It is a direct application of Corollary 3.4. Indeed, if $\underline{\ell}$ and $\bar{\ell}'$ are finite, then

$$\liminf_{n \rightarrow \infty} \frac{a_{b_n}}{n \cdot b_n} \leq \liminf_{n \rightarrow \infty} \frac{a_{b_n}}{b_n \cdot \vartheta(b_n)} \cdot \limsup_{n \rightarrow \infty} \frac{\vartheta(b_n)}{n} \leq \underline{\ell} \cdot \bar{\ell}';$$

and if $\bar{\ell}$ and $\underline{\ell}'$ are finite, then

$$\liminf_{n \rightarrow \infty} \frac{a_{b_n}}{n \cdot b_n} \leq \limsup_{n \rightarrow \infty} \frac{a_{b_n}}{b_n \cdot \vartheta(b_n)} \cdot \liminf_{n \rightarrow \infty} \frac{\vartheta(b_n)}{n} \leq \bar{\ell} \cdot \underline{\ell}'.$$

□

As witnessed by the results proved above, if A has zero density but still it is “large” enough, then its set of distances intersect sets of distances of really “sparse” sets B . We give below two examples to illustrate this phenomenon.

Example 3.6. Let $P = \{p_n\}$ be the set of prime numbers, and let $B = \{2^n\}$ be the set of powers of 2. By the *Prime Number Theorem*,

$$\lim_{n \rightarrow \infty} \frac{p_n}{n \cdot \log n} = 1.$$

Since $(\log 2^n)/n = \log 2 < 1$, by the previous corollary we can conclude that for every k , there exist numbers of the form $2^m - 2^n$ which are the distance of at least k -many pairs of primes. Actually, there exist infinitely many such numbers, since the function $(n, m) \mapsto 2^m - 2^n$ is 1-1; indeed, first pick $2^{n_1} - 2^{m_1} \in R_k(P) \cap \Delta(B)$, then consider $B^{(1)} = B \setminus \{2^{n_1}, 2^{n_2}\}$ and pick $2^{n_2} - 2^{m_2} \in R_k(P) \cap \Delta(B^{(1)})$, and so forth.

Example 3.7. Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty \quad \text{and} \quad \log b_n \ll n^{1-\varepsilon} \quad \text{for some } \varepsilon > 0.$$

Then $R_k(A) \cap \Delta(B)$ is infinite for all k .

Proof. If we let $\vartheta(n) = (\log n)^{\frac{1}{1-\varepsilon}}$, the hypotheses imply that

$$\liminf_{n \rightarrow \infty} \frac{a_n}{n \cdot \vartheta(n)} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\vartheta(b_n)}{n} = \left(\limsup_{n \rightarrow \infty} \frac{\log b_n}{n^{1-\varepsilon}} \right)^{\frac{1}{1-\varepsilon}} = 0,$$

and the desired intersection property follows by Corollary 3.5. □

E.g., if $A = \{a_n\}$ is such that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$, then for every exponent $\alpha < 1$ and for every k , there exist infinitely many numbers of the form $\lfloor 10^{n^\alpha} \rfloor - \lfloor 10^{m^\alpha} \rfloor$, everyone of which is the distance of at least k -many different pairs of elements of A .

Let us now focus on powers of n .

Theorem 3.8. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that, for all sufficiently large n ,*

$$a_n \leq K \cdot n^{1+\alpha} \quad \text{and} \quad b_n \leq M \cdot n^{1+\beta}.$$

- (1) *If $\alpha < 1$ and $\beta < 1/\alpha$ then $R_k(A) \cap \Delta(B)$ is infinite for all k .*
- (2) *If $\alpha < 1$ and $\beta = 1/\alpha$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k whenever $K^\beta M < \frac{\alpha}{(1+\alpha)^{\beta+1}}$.*
- (3) *If $\alpha = \beta = 1$ then $\Delta(A) \cap \Delta(B) \neq \emptyset$ whenever $KM < \frac{1}{4}$.*

Proof. Notice first that, without loss of generality, we can assume $n \lll a_n$, and hence $\alpha > 0$. Indeed, otherwise $\bar{d}(A) > 0$, and the thesis is proved by Corollary 2.6.

(1). The thesis follows from (2) of Theorem 3.1 since $a_n \lll n^2$ and for every constant $c > 1$ one has that

$$\liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \leq \lim_{n \rightarrow \infty} \frac{M \cdot (c \cdot K \cdot n^\alpha)^\beta}{n} = \lim_{n \rightarrow \infty} M \cdot c^\beta \cdot K^\beta \cdot \frac{n^{\alpha\beta}}{n} = 0.$$

(2). We use (1) of Theorem 3.1. Given a constant $c > 1$, under our hypotheses one has that

$$\liminf_{n \rightarrow \infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \leq M \cdot c^\beta \cdot K^\beta.$$

Now,

$$M \cdot c^\beta \cdot K^\beta < 1 - \frac{1}{c} \iff M \cdot K^\beta < \frac{c-1}{c^{\beta+1}},$$

and the greatest possible value of the last expression is attained when $c = 1 + \alpha$, namely $\frac{\alpha}{(1+\alpha)^{\beta+1}}$, as one can directly verify.

(3). Fix a constant $c > 0$. For every given n , let $N = n$ and $\nu = \tau(n) = \lfloor c \cdot \sqrt{KM} \cdot n \rfloor$. By Lemma 2.3, there exists an element $x_n \in \Delta(B)$ such that

$$\begin{aligned} |A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| &\geq \frac{n^2}{a_n + b_{\tau(n)}} - \frac{n}{\tau(n)} \geq \\ &\geq \frac{n^2}{Kn^2 + Mc^2 \cdot \frac{K}{M} \cdot n^2} - \frac{n}{\lfloor c \cdot \sqrt{\frac{K}{M}} \cdot n \rfloor} = \frac{1}{K} \cdot \left(\frac{1}{1+c^2} - \frac{\sqrt{KM}}{c} \cdot \psi(n) \right) \end{aligned}$$

where $\psi(n) = \frac{c \cdot \sqrt{KM} \cdot n}{\lfloor c \cdot \sqrt{KM} \cdot n \rfloor} \rightarrow 1$ as $n \rightarrow \infty$. So, the last quantity above is positive for all sufficiently large n if and only if $\sqrt{KM} < \frac{c}{1+c^2}$. Now, it is easily checked that the greatest possible value of the latter expression is $1/2$, which is attained when $c = 1$. This means that if $KM < 1/4$ then there exist elements $x_n \in \Delta(A) \cap \Delta(B)$, *i.e.* the thesis. \square

Example 3.9. Let $A = \{\lfloor K \cdot n \sqrt{n} \rfloor\}$ and $B = \{n^3\}$. If $K^2 \cdot M < 4/27$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k . Indeed, we can apply (2) of the theorem above, where $1/\alpha = \beta = 2$.

4. A VARIANT OF KHINTCHINE'S THEOREM

In this final section we exploit further consequences of Lemma 2.3 and prove a result for a class of zero density sets that resembles *Khintchine's Recurrence Theorem*.

Let us first introduce some notation. For sets $A \subseteq \mathbb{N}$, we write $\mathfrak{d}(A)_n$ to denote the relative density of A on the interval $[1, n]$, *i.e.*

$$\mathfrak{d}(A)_n = \frac{|A \cap [1, n]|}{n}.$$

As already pointed out, the limit superior given by the upper asymptotic density is attained along intervals of the form $[1, a_n]$; so one has

$$\bar{\mathfrak{d}}(A) = \limsup_{n \rightarrow \infty} \mathfrak{d}(A)_{a_n} = \limsup_{n \rightarrow \infty} \frac{n}{a_n}.$$

Theorem 4.1. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and assume that*

$$\liminf_{n \rightarrow \infty} \frac{a_n \cdot b_n}{n^2 \cdot b_n} = l < \frac{1}{2}.$$

Then there exists a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ of elements of $\Delta(B)$ such that

$$\limsup_{n \rightarrow \infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) \geq 1 - 2l > 0.$$

Proof. For every n , let $\sigma(n) = n \cdot b_n$, and apply Lemma 2.3 with $h = 1$ to the sets $A_{\sigma(n)} = \{a_1 < \dots < a_{\sigma(n)}\}$ and $B_n = \{b_1 < \dots < b_n\}$. We obtain the existence of an element $x_n \in \Delta(B_n) \subseteq \Delta(B)$ such that:

$$\begin{aligned} |A \cap (A + x_n) \cap [1, a_{\sigma(n)}]| &\geq |A \cap (A + x_n) \cap [1, a_{\sigma(n)} + b_n]| - b_n \geq \\ &\geq |A_{\sigma(n)} \cap (A_{\sigma(n)} + x_n)| - b_n \geq \frac{\sigma(n)^2}{a_{\sigma(n)}} \cdot \frac{1 - \frac{a_{\sigma(n)} + b_n}{\sigma(n) \cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} - b_n. \end{aligned}$$

By combining, one gets

$$\begin{aligned} \frac{\mathfrak{d}(A \cap (A + x_n))_{a_{\sigma(n)}}}{(\mathfrak{d}(A)_{a_{\sigma(n)}})^2} &= \frac{|A \cap (A + x_n) \cap [1, a_{\sigma(n)}]|}{\frac{\sigma(n)^2}{a_{\sigma(n)}}} \geq \\ &\geq \frac{1 - \frac{a_{\sigma(n)} + b_n}{\sigma(n) \cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} - \frac{a_{\sigma(n)} \cdot b_n}{\sigma(n)^2}. \end{aligned}$$

Now notice that:

- $\liminf_{n \rightarrow \infty} \frac{a_{\sigma(n)}}{\sigma(n) \cdot n} = \liminf_{n \rightarrow \infty} \frac{a_n \cdot b_n}{n^2 \cdot b_n} = l$;
- $\lim_{n \rightarrow \infty} \frac{b_n}{\sigma(n) \cdot n} = \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$;
- $\lim_{n \rightarrow \infty} \frac{b_n}{a_{\sigma(n)}} = \lim_{n \rightarrow \infty} \frac{n \cdot b_n}{a_n \cdot b_n} \cdot \frac{1}{n} \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0$;
- $\liminf_{n \rightarrow \infty} \frac{a_{\sigma(n)} \cdot b_n}{\sigma(n)^2} = \liminf_{n \rightarrow \infty} \frac{a_n \cdot b_n}{n^2 \cdot b_n} = l$.

By considering the inequalities proved above, and by passing to the limit superiors as n goes to infinity, we finally get:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) &\geq \limsup_{n \rightarrow \infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_{\sigma(n)}}}{(\mathfrak{d}(A)_{a_{\sigma(n)}})^2} \right) \geq \\ &\geq \limsup_{n \rightarrow \infty} \left(\frac{1 - \frac{a_{\sigma(n)}}{\sigma(n) \cdot n} - \frac{b_n}{\sigma(n) \cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} - \frac{a_{\sigma(n)} \cdot b_n}{\sigma(n)^2} \right) = 1 - 2l > 0. \end{aligned}$$

□

Corollary 4.2. *Let $A = \{a_n\}$ be an infinite set of natural numbers. If $a_n \lll n^{3/2}$ then there exists a sequence of shifts $\langle x_n \mid n \in \mathbb{N} \rangle$ such that*

$$\limsup_{n \rightarrow \infty} \left(\frac{\frac{|A \cap (A + x_n) \cap [1, n]|}{n}}{\left(\frac{|A \cap [1, n]|}{n} \right)^2} \right) \geq 1.$$

Proof. Let $B = \mathbb{N}$. Then the previous theorem applies where $l = 0$, and the thesis easily follows. □

Similarly as Corollary 3.5 is derived from Theorem 3.4, one proves the following property as a straight consequence of Theorem 4.1.

Corollary 4.3. *Assume that, for a suitable $\vartheta : \mathbb{N} \rightarrow \mathbb{R}^+$, the infinite sets of natural numbers $A = \{a_n\}$ and $B = \{b_n\}$ satisfy*

$$\limsup_{n \rightarrow \infty} \frac{a_n}{n \cdot \vartheta(n)} = l_1 < \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\vartheta(n \cdot b_n)}{n} = l_2 < \infty$$

where $l_1 \cdot l_2 < 1/2$. Then there exists a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ of elements of $\Delta(B)$ such that

$$\limsup_{n \rightarrow \infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) \geq 1 - 2l_1 l_2 > 0.$$

Proof. Theorem 4.1 applies, since

$$\liminf_{n \rightarrow \infty} \frac{a_n \cdot b_n}{n^2 \cdot b_n} \leq \limsup_{n \rightarrow \infty} \frac{a_n \cdot b_n}{n \cdot b_n \cdot \vartheta(n \cdot b_n)} \cdot \liminf_{n \rightarrow \infty} \frac{\vartheta(n \cdot b_n)}{n} \leq l_1 l_2 < \frac{1}{2}.$$

□

To illustrate the use of the above corollary, let us see a property that holds for all sets $A = \{a_n\}$ having the same asymptotic size as the set of primes.

Proposition 4.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that*

$$\lim_{n \rightarrow \infty} \frac{a_n}{n \log n} = 1 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\log b_n}{n} = 0.$$

Then for every $\varepsilon > 0$ there exist infinitely many n and elements $x_n \in \Delta(B)$ such that

$$|A \cap (A + x_n) \cap [1, a_n]| \geq \frac{n}{\log n} \cdot (1 - \varepsilon).$$

Proof. Let $\vartheta(n) = \log n$. By the hypotheses,

$$\lim_{n \rightarrow \infty} \frac{a_n}{n \cdot \vartheta(n)} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\vartheta(n \cdot b_n)}{n} = \lim_{n \rightarrow \infty} \frac{\log n + \log b_n}{n} = 0.$$

So, the previous corollary applies, and we get the existence of elements $x_n \in \Delta(B)$ such that

$$\limsup_{n \rightarrow \infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) \geq 1.$$

Now notice that

$$\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} = |A \cap (A + x_n) \cap [1, a_n]| \cdot \frac{a_n}{n^2}.$$

So, for every $\delta > 0$, there exist infinitely many n that satisfy

$$|A \cap (A + x_n) \cap [1, a_n]| \cdot \frac{a_n}{n^2} \geq 1 - \delta.$$

By our hypothesis on $\{a_n\}$, we know that $\frac{n \cdot \log n}{a_n} \geq 1 - \delta$ for all sufficiently large n , and so we can conclude that there exist infinitely many n and elements $x_n \in \Delta(B)$ such that:

$$|A \cap (A + x_n) \cap [1, a_n]| \geq \frac{n^2}{a_n} \cdot (1 - \delta) = \frac{n}{\log n} \cdot \frac{n \log n}{a_n} \cdot (1 - \delta) \geq \frac{n}{\log n} \cdot (1 - \delta)^2.$$

The proof is completed by choosing δ in such a way that $(1 - \delta)^2 \geq 1 - \varepsilon$. \square

Example 4.5. Let $P = \{p_n\}$ be the set of prime numbers. Then, for any given $\varepsilon > 0$, there exist arbitrarily large n such that one finds “nearly” $(n/\log n)$ -many pairs of primes $p, p' \leq p_n$ which have a common distance $p - p' = d$. Moreover, such a distance d can be taken to belong to any prescribed set of distances $\Delta(B)$, provided $B = \{b_n\}$ is not too sparse in the precise sense that $\log b_n \lll n$ (e.g., one can take $b_n = \lfloor 10^{\frac{n}{\log n}} \rfloor$).

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